INTERNATIONAL MATHEMATICS
TOURNAMENT OF TOWNS

JUNIOR PAPER: YEARS 8,9,10

Tournament 41, Northern Autumn 2019 (O Level)

Note: Each contestant is credited with the largest sum of points obtained for three problems.

1. An illusionist lays a full deck of 52 cards in a row and tells spectators that 51 cards will be taken away step by step with only the Three of Clubs remaining on the table. On each step some spectator tells the illusionist a number so that a card lying on the place with this number in the row is taken away. However, the illusionist makes his own decision from which side of the row, left or right, he should count that number from to take the card away. For which initial positions of the Three of Clubs can the illusionist guarantee the success of his trick for sure? (4 points)

2. Let ω be a circle with the centre O and two different points A and C on ω. For any point P on ω distinct from A and C let X and Y be the midpoints of AP and CP respectively. Futhermore, let H be the point where the altitudes of triangle OXY meet. Prove that the position of the point H does not depend on the choice of P. (4 points)

3. There is a row of 100 squares each containing a counter. Any 2 neighbouring counters can be swapped for 1 dollar and any 2 counters that have exactly 3 counters between them can be swapped for free. What is the least amount of money that must be spent to rearrange the counters in reverse order? (4 points)

4. Let $a_1, \ldots, a_{1000}$ be given integers. Their squares $a_1^2, \ldots, a_{1000}^2$ are written around a circle. It is known that the sum of any 41 consecutive numbers on this circle is a multiple of 41^2. Is it necessarily true that each of the integers $a_1, \ldots, a_{1000}$ is a multiple of 41? (5 points)

5. Vasya has an unlimited supply of bricks of size $1 \times 1 \times 3$ and L-shape bricks, both made of three cubes of size $1 \times 1 \times 1$. Vasya completely filled a box of size $m \times n \times k$ with these bricks, where m, n and k are integers greater than 1. Prove that he can completely fill the same box using L-shape bricks only. (5 points)

O Level Junior Paper Solutions

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Note marking notations and their suggested application are provided at the end.

1. The illusionist can guarantee the success of his trick when the Three of Clubs is placed at either end of the row of 52 cards. Indeed, the illusionist can be forced to take away the Three of Clubs only if it is a central (symmetrical with respect to both ends of the row) card at some step. Otherwise he always has the opportunity to remove another card. If the Three of Clubs is placed at either end of the row of 52 cards, it can only become a central card on the very last turn, allowing the illusionist to guarantee the success of his trick.

We claim that for any other position the spectators can prevent the success of the trick. Suppose that the Three of Clubs is placed at an interior position of the row of 52 cards, i.e. not at either end. Here are two possible strategies for spectators.

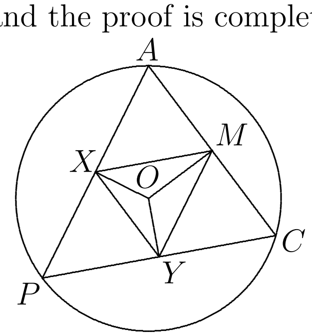
Strategy 1. The spectators keep nominating numbers that match interior positions. This guarantees that the illusionist cannot remove a card from either end. When there only 3 cards remaining in the row, the Three of Clubs is necessarily the middle card. The spectators should then nominate number 2 and force the illusionist to remove the Three of Clubs.

Strategy 2. The spectators always nominate the number of the position of the Three of Clubs. Then the illusionist is forced to remove the card in the mirror symmetric position, decreasing the larger of the distances of the Three of Clubs from the ends of the row. Thus, eventually the Three of Clubs is the same distance from both ends, which is to say it is the middle card, so that the illusionist is forced to remove it while it is still an interior card.

2. Recall that the common intersection point of the altitudes of a triangle, is its orthocentre. Thus H is the orthocentre of triangle OXY.

Let M be the midpoint of AC. We claim that M and H coincide. Indeed, since O and M are each equidistant from A and C, and the locus of points equidistant from two points is the perpendicular bisector of the two points, OM is perpendicular to AC.

Since X and Y are the midpoints of AP and CP respectively, XY (as a middle line) is parallel to AC, and, therefore, perpendicular to OM. Similarly, OX is perpendicular to YM and OY is perpendicular to XM. Thus, OM, YM and XM are altitudes of triangle OXY, and hence $M = H$, the orthocentre of triangle OXY. Since, the position of $M = H$ as the midpoint of AC, is determined only by A and C, it does not depend on the choice of P and the proof is complete.



3. This is a "discrete optimisation" problem. We will first show that the amount of money that must be spent to rearrange the counters in reverse order is bounded below by some value B, and then we show, by explicit example, that B can be attained, where $B = \$50$. Thus we will have shown that the least amount of money that must be spent to rearrange the counters in reverse order is \$50.

Indeed, for reverse order each counter must change the parity of its position. Since the parity of the position of 2 counters that have exactly 3 counters between them is the same, swapping them leaves the parity of their positions unchanged, whereas the swap of 2 neighbouring counters changes the parity of the position of both counters. Hence, at least \$50 must be spent in order to change the parity of the position of all 100 counters (\$1 for each swap of 2 neighbouring counters).

Now we provide an example where exactly \$50 is spent.

Colour all 100 squares of the row according to the following pattern of 4 colours,

$$abcd\,abcd\,abcd\dots abcd.$$

Note that we can we swap two counters in the consecutive squares of the same colour for free. So we can arrange the counters in the squares of the same colour in any order we wish without spending money. First, however, we swap the counters in all pairs bc and in all pairs da, to obtain

$$a\,cbad\,cbad\dots cbad\,cbd.$$

To do so we need to spend $\$49 = 24 \times 2 + 1$.

Next using free swaps, we propagate the leftmost counter of colour a with d counters until it is in the 97th position, to obtain

$$d\,cbad\,cbad\dots cbaa\,cbd,$$

and swap (also for free) the rightmost counter of colour d with the counter of colour a in the 96th position to obtain

$$d\,cbad\,cbad\dots cbda\,cba.$$

Now we expend the 50th dollar swapping the d and a in the 96th and 97th places to obtain

$$d\,cbad\,cbad\dots cbad\,cba.$$

So now we have the colouring that corresponds to counters being in reverse order, and we may swap counters of the same colour for free until the counters are actually finally in the reverse of the original order.

4. Yes, it is necessarily true that each of the integers $a_1, \dots, a_{1000}$ is a multiple of 41.

Indeed, taking 42 consecutive numbers $x_1, x_2, \dots, x_{42}$ around the circle, and considering their two 41 consecutive element subsequences, we have

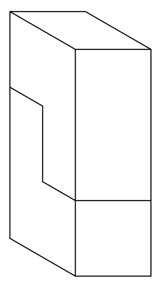
$$x_1 + x_2 + \dots + x_{41} \equiv 0 \equiv x_2 + x_3 + \dots + x_{42} \pmod{41^2}$$
$$x_1 \equiv x_{42} \pmod{41^2}$$

As a consequence, $a_i^2 \equiv a_j^2 \pmod{41^2}$, if $i \equiv j \pmod{41}$.

Since 1000 and 41 are coprime, it follows that all a_i^2 have the same residue r modulo 41^2. Hence, the sum of any 41 consecutive numbers around the circle will be congruent to $41r$ modulo 41^2 and is also a multiple of 41^2, which means that r is a multiple of 41, i.e. 41 divides a_i^2 for each i. Since 41 is prime, it follows that each a_i is a multiple of 41, and the proof is complete.

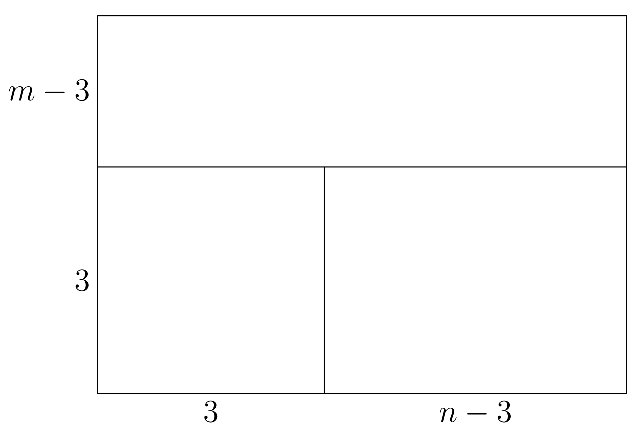
5. Since both bricks are made of three cubes of size $1 \times 1 \times 1$ and Vasya can completely fill a box of size $m \times n \times k$ with these bricks, the product mnk must be divisible by 3. Hence, one of the three factors must be divisible by 3. Without loss of generality assume the height k is divisible by 3. Then, it is sufficient to show that Vasya can completely fill a box of the size $m \times n \times 3$ using L-shape bricks only. Now we have two cases.

Case 1: m or n is even. Without loss of generality m is even. Observe that Vasya can form a $1 \times 2 \times 3$ block by putting together 2 L-shape bricks.

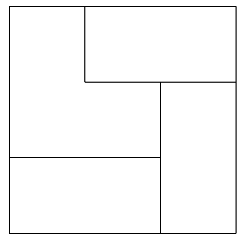


The base of this block has the shape of a (2×1) domino, and our problem is reduced to fitting such dominos into the mn area of the base of the box. Vasya can do so by forming an $m \times 1$ strip of $m/2$ dominos lengthwise, and then placing n such strips side by side. Thus an $m \times n \times 3$ box can be filled with L-shape bricks only, in this case.

Case 2: m and n are both odd. Then since $m, n, k > 1$, we have $m, n \geq 3$. Then the base of the box can be divided into a square of size 3×3 and possibly empty rectangles of size $(m-3) \times n$ and $3 \times (n-3)$:



Since $m-3$ and $n-3$ are even, we can fill each of the sub-boxes of height 3 with bases $(m-3) \times n$ and $3 \times (n-3)$, according to the strategy of Case 1, and, of course, in each of the cases where $m = 3$ or $n = 3$, where there is an empty sub-box, there is nothing to do. And so we have reduced the problem to filling a $3 \times 3 \times 3$ box with L-shape bricks, whose base is as shown below, divided into an L-shape and three domino shapes.



On each domino-shaped area, Vasya can place a $1 \times 2 \times 3$ block formed from 2 L-shape bricks, and on the L-shape area, Vasya can stack three L-shape bricks. Thus an $m \times n \times 3$ box can be filled with L-shape bricks only, in this case, also.

And so the proof is complete.

Preferred Marking Notations

Please use the following notations when marking scripts.

Notation	Meaning
$+$	solution is correct.
$\pm$	solution has a serious but easily fixable hole.
$\mp$	solution is incorrect, but contains some correct steps.
$-$	solution is incorrect.
0	no solution is present at all.
$+.$	variation of "$+$" for smaller hole than "$\pm$".
$-.$	variation of "$-$" for smaller achievements than "$\mp$".
$+/2$	used in specific cases when there is a correct important idea given, but is not developed enough to consider the problem to be solved. It may be used when the problem is inherently divided into two halves and one of them is solved.
$+!$	an unusual and special achievement such as a beautifully succinct or elegant proof, or a more general result.